Fluctuations of Random Noise Power*

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The probability distribution of the power, y, of a sample of Gaussian noise of time duration T is considered. Some general theory is presented along with curves for the cumulative distribution and probability density of y for several different power spectra and values of T.

I. INTRODUCTION

A random quantity of interest in many communication and detection systems is the average power,

$$y = \frac{1}{T}\int_{-T/2}^{T/2} N^2(t)\, dt, \tag{1}$$

of a sample of finite time duration, T, of a Gaussian noise, $N(t)$. This quantity has been discussed in some detail by Rice in his classic paper[1] where he obtains expressions for the first few moments of y and an approximate probability density function.

In this paper the exact probability density function, $f(y)$, and the cumulative distribution function, $F(y)$, of the average power are computed for a number of ergodic Gaussian noises and for a number of values of T. The results are presented as a series of curves which are discussed in the next section. It is hoped that they will be of use to those designing specific systems.

II. SUMMARY OF COMPUTATIONAL RESULTS

Fig. 1 shows the probability density function, $f(y)$, for the random variable y of equation (1) when $N(t)$ has mean zero and power spectrum

$$w(f) = \frac{2\alpha}{\alpha^2 + 4\pi^2 f^2}, \quad -\infty \leqq f \leqq \infty .$$

Noise with this spectrum will be referred to as RC noise (see 5.1).

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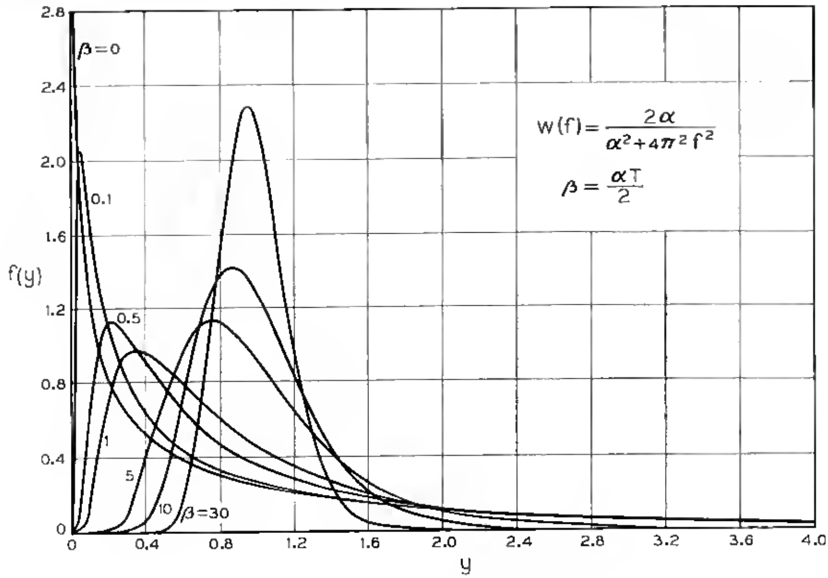

Fig. 1 — Probability density, $f(y)$, for RC noise.

The curves are labelled by values of $\beta = \alpha T/2$. The curve marked $\beta = 0$ is the probability density function for $y = N^2(t)$. Fig. 2 shows the corresponding cumulative distribution functions, $F(y)$.

For any $\beta > 0$, as y approaches zero, $f(y)$ and $F(y)$ approach zero more rapidly than any power of y.

As β becomes large, the density function $f(y)$ peaks up around unity which is the average power of $N(t)$. The variance of y is given by $(2\beta)^{-2}[4\beta - 1 + e^{-4\beta}]$. It approaches zero for large β like β^{-1}.

Figs. 3, 4 and 5 show $f(y)$ when $N(t)$ has mean zero and power spectrum

$$w(f) = \frac{2Q}{w_0} \frac{\omega^2}{\omega^2 + \left(\frac{Q}{\omega_0}\right)^2 (\omega^2 - \omega_0^2)^2}, \tag{2}$$

$$\omega = 2\pi f, \quad -\infty \leqq f \leqq \infty.$$

Noise with this spectrum will be referred to as RLC Noise (see 5.2). The figures are respectively for the cases $Q = 1$, 10 and 100. The curves are labelled by values of $s = \omega_0 T$. The curves marked $s = 0$ are the density function for $y = N^2(t)$. The corresponding cumulative density

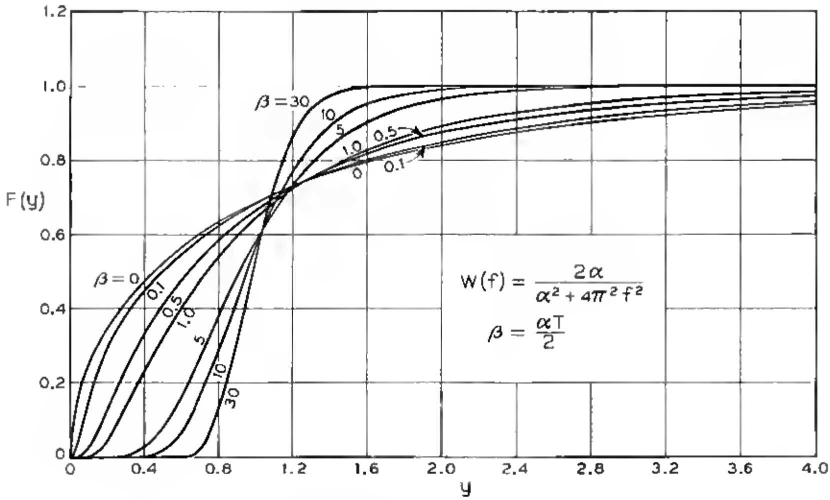

Fig. 2 — Cumulative distribution, $F(y)$, for RC noise.

functions, $F(y)$, are shown on Figs. 6, 7 and 8. The spectra for $Q = 1$, 10 and 100 are plotted on Fig. 9.

For any $s > 0$ and for any finite $Q > 0$, as y approaches zero, both $f(y)$ and $F(y)$ approach zero more rapidly than any power of y.

For any fixed Q, as s becomes large, the density function $f(y)$ peaks up around unity which is the average power of $N(t)$. The variance of y is given by

$$\begin{aligned} \sigma^2 &= \frac{2}{\tau^2}\left[\tau - 1 + e^{-\tau} + \frac{2e^{-\tau}}{4Q^2 - 1}\sin^2\frac{\tau}{2}\sqrt{4Q^2 - 1}\right], \\ \tau &= \frac{s}{Q}. \end{aligned} \tag{3}$$

For fixed Q, it approaches zero for large s like $2Q/s$.

If, however, $s = \omega_0 T$ is held fixed and Q is permitted to increase, Figs. 3, 4 and 5 show that $f(y)$ becomes less concentrated; that is, with fixed integration time and fixed resonant frequency, fluctuations in power become more pronounced as the relative width of the spectral peak is decreased. Indeed, one has

$$\lim_{Q\to\infty} \sigma^2 = 1 + \frac{\sin^2 s}{s^2},$$

so that

$$\lim_{s\to\infty} \lim_{Q\to\infty} \sigma^2 = 1,$$

whereas, as already noted,

$$\lim_{Q\to\infty} \lim_{s\to\infty} \sigma^2 = 0.$$

In the limit $Q = \infty$, the Gaussian noise can be taken to be the single frequency ensemble $N(t) = a \cos \omega_0 t + b \sin \omega_0 t$, where a and b are independent normal variates with mean zero and variance unity. The density for y in this case is

$$f(y) = \sec \varphi e^{-y \sec^2 \varphi} J_0(iy \tan \varphi \sec \varphi)$$

where $\sin \varphi = \sin s/s$ and J_0 is the usual Bessel function (see Appendix 1). This density is plotted for several values of s in Fig. 10. It is to be noted that this limiting noise, although stationary, is not ergodic. It is this fact that causes the variance of y to be bounded away from zero as $s \to \infty$. Quite generally, if $N(t)$ has a purely continuous spectrum, the variance of y will approach zero as the integration time becomes infinite. If the spectrum of $N(t)$ has line components, this will not be the case.

It is not difficult to give a qualitative argument as to why power fluctuations in a fixed time interval increase as the power spectrum becomes

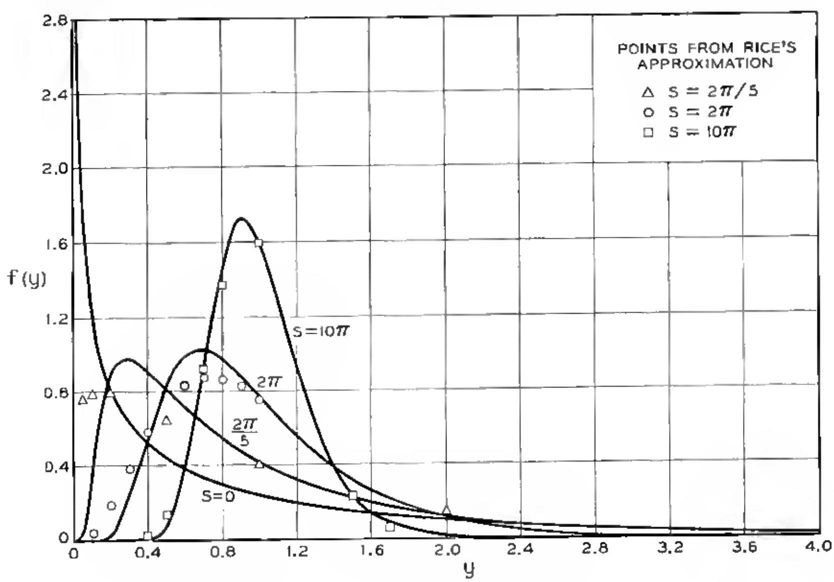

Fig. 3 — Probability density, $f(y)$, for RLC noise, $Q = 1.0$, $s = \omega_0 T$.

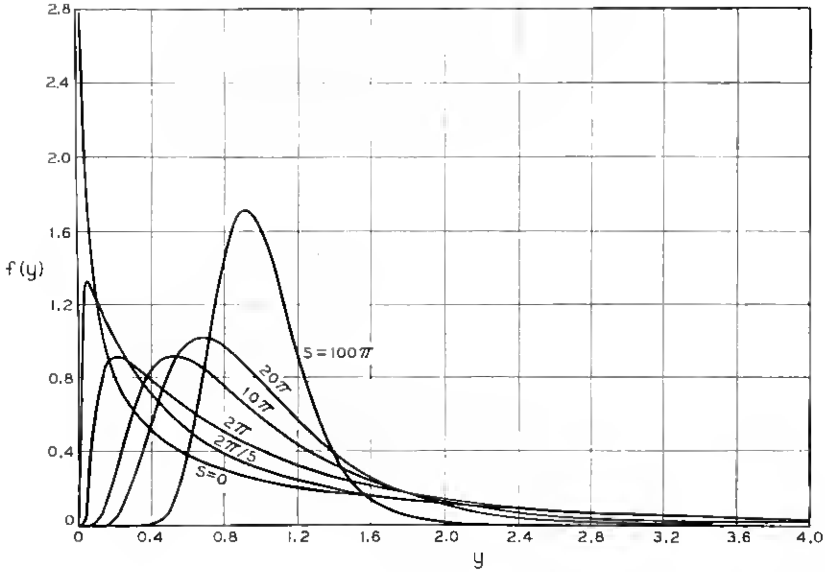

Fig. 4 — Probability density, $f(y)$, for RLC noise, $Q = 10$, $s = \omega_0 T$.

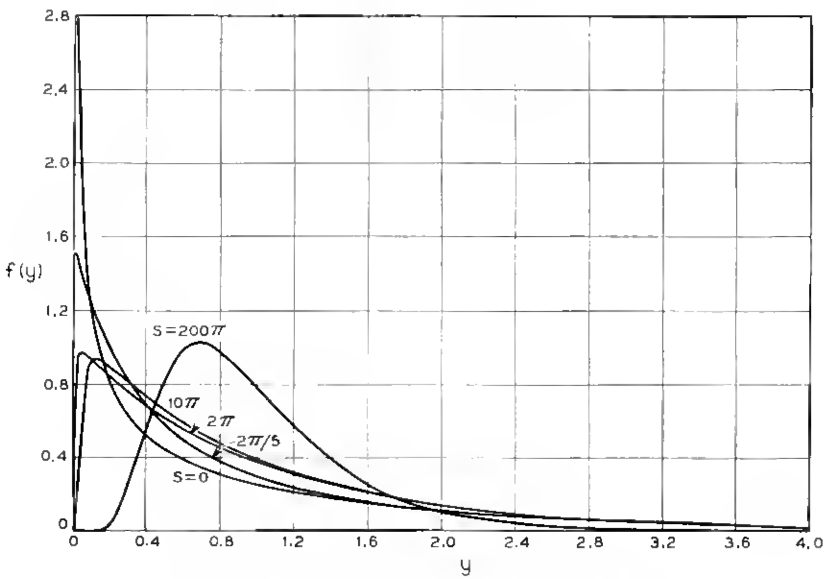

Fig. 5 — Probability density, $f(y)$, for RLC noise, $Q = 100$, $s = \omega_0 T$.

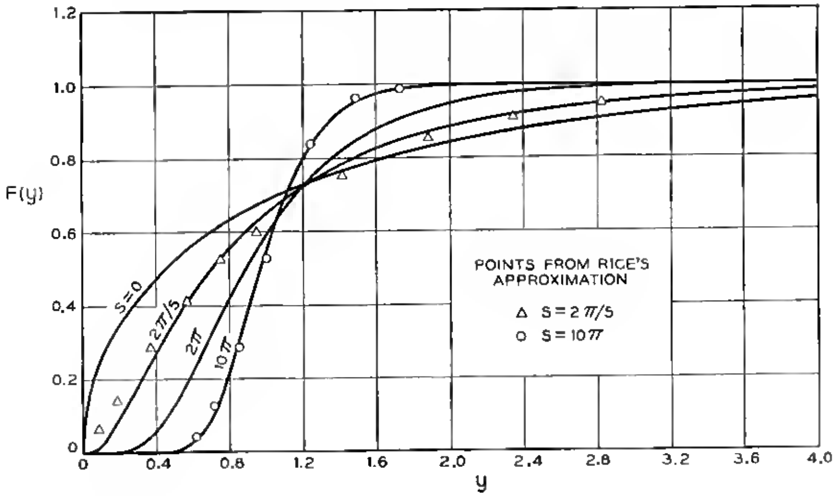

Fig. 6 — Cumulative distribution, $F(y)$, for RLC noise, $Q = 1.0$, $s = \omega_0 T$.

more peaked. Noise with the power spectrum (2) can be thought of as the noise voltage produced across the resistor in a series RLC circuit when the applied voltage to the circuit is white Gaussian noise. The larger the Q of the circuit, the more it tends to "ring" in response to an impulse input; i.e., the longer the transients persist. An atypical excur-

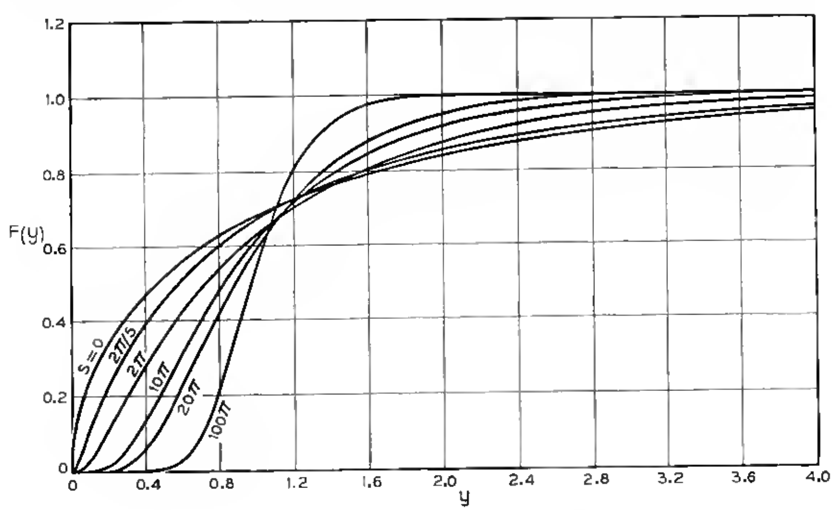

Fig. 7 — Cumulative distribution, $F(y)$, for RLC noise, $Q = 10$, $s = \omega_0 T$.

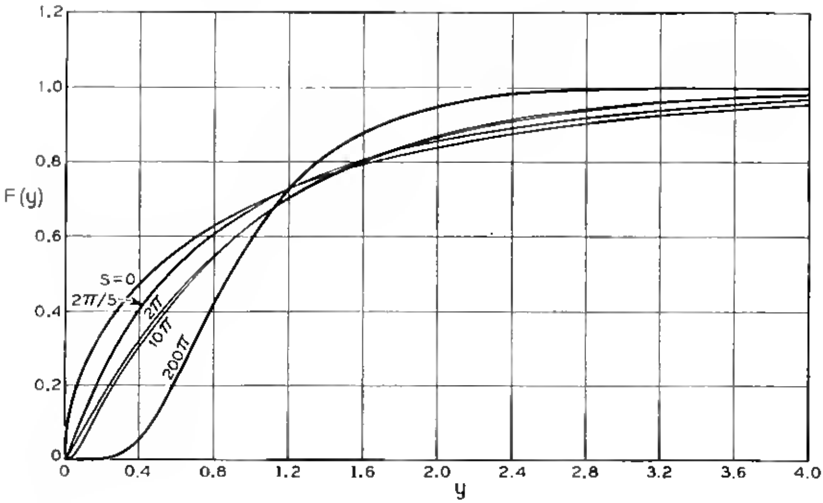

Fig. 8 — Cumulative distribution, $F(y)$, for RLC noise, $Q = 100$, $s = \omega_0 T$.

sion of the input voltage will therefore have a longer lasting effect in the output of a circuit with a large Q than in a circuit with a small Q. To obtain the same variance in power, then, the integration time must be longer in the circuit with the large Q value. It would seem reasonable to expect this argument to apply for any peaked spectrum, not solely for (2).*

If the Q of the spectrum (2) is increased, how much must the integration time be increased to maintain roughly the same power fluctuations? From (3), it is seen that for large Q, σ^2 is approximately $2\tau^2[\tau - 1 + e^{-\tau}]$, i.e., a function of

$$\tau = \frac{s}{Q} = \frac{\omega_0}{Q} T$$

alone. Now Q measures the relative sharpness of the spectral peak, so that ω_0/Q is a measure of the absolute widtb of the peak in radians/sec. As a rough rule, then, power measurements from different members of the family (2) will have the same fluctuations if their products "integration time" times "absolute spectral bandwidth" are the same. Fig. 11 shows σ^2 as a function of τ for $Q = 1$, 10, and 100. That τ is a good measure of the fluctuation in power can also be seen by comparing the f curves of equal τ value in Figs. 3, 4 and 5. They are almost identical.

* It seems to be very difficult to make any other qualitative statements regarding the relation between the shape of the noise spectrum and the density function for y.

On Fig. 11 the variance of y for bandpass noise with spectrum

$$w(f) = \begin{cases} \dfrac{1}{4\delta}, & \mid f \pm f_0 \mid \leqq \delta \\ 0, & \mid f \pm f_0 \mid > \delta \end{cases}$$

is plotted versus

$$\tau = \frac{s}{Q_b} = \frac{2\pi f_0 t}{Q_b}.$$

Here the Q of a bandpass circuit is defined by

$$Q_b = \pi \frac{f_0}{2\delta}$$

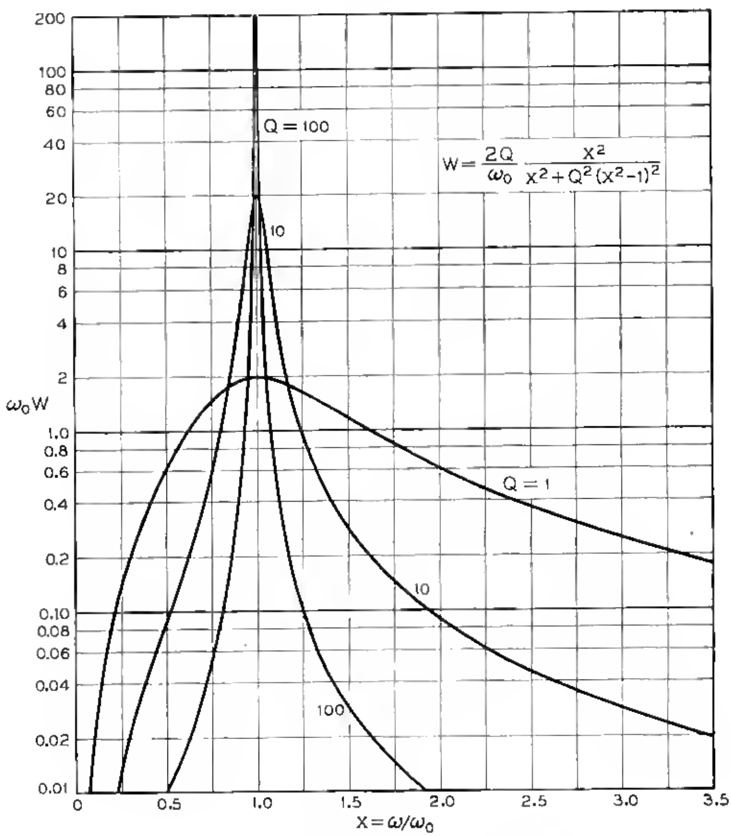

Fig. 9 — RLC spectra, $Q = 1, 10, 100$.

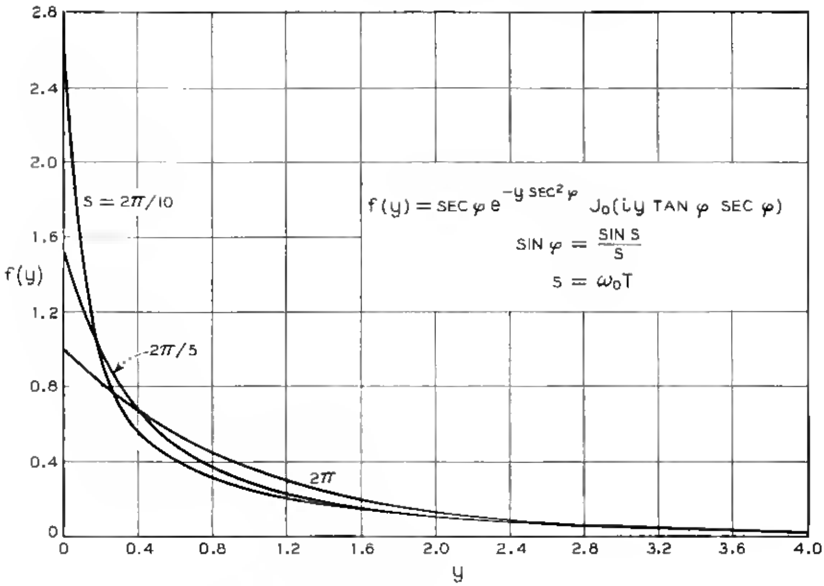

Fig. 10 — Probability density, $f(y)$, for RLC noise, $Q = \infty$.

and measures the relative width of the spectrum. This definition of Q_b causes the σ^2 curves of this noise power to agree asymptotically with those of the RLC noise power; namely $\sigma^2 \sim 2/\tau$ in both cases. Again, when it is not too small, τ seems to be a good measure of power fluctuations. The variance in this bandpass case is given by

$$\sigma_b^2 = 4 \int_0^1 (1 - y) \left[\frac{\cos Q_b \tau y \sin \dfrac{\pi \tau y}{2}}{\dfrac{\pi \tau y}{2}} \right]^2 dy$$

which can be readily evaluated in terms of Si and Ci functions. The curve for $Q_b = 100$ coincides so closely with the curve for $Q_b = 10$ it could not be shown on Fig. 11.

The asymptotic agreement of the variance of noise power for bandpass and RLC noise permitted defining the Q of the bandpass circuit as $Q_b = \pi(f_0/2\delta)$. These same considerations suggest defining the bandwidth W of the RLC spectrum by $W = \omega_0/2Q$. For, in the bandpass case, $\tau = 2(2\delta)T$ which is $2T$ times the bandwidth of the spectrum. For the RLC noise, $\tau = \omega_0 T/Q = 2T(\omega_0/2Q)$, whence the definition of W follows.

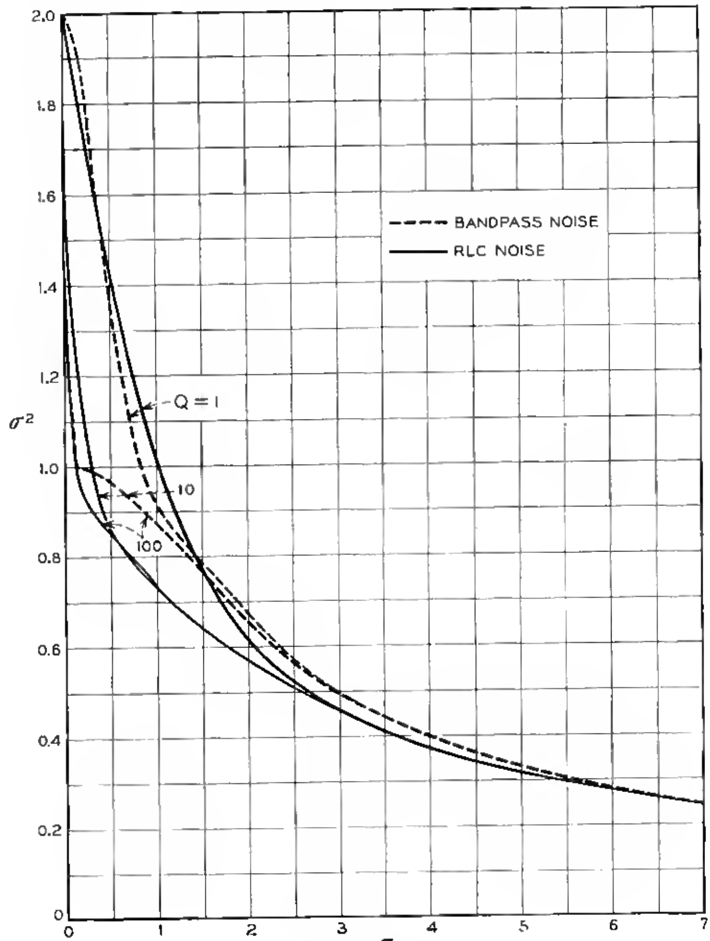

Fig. 11 — σ^2 for RLC noise and bandpass noise, $Q = 1, 10, 100$.

The curves shown in Figs. 1–8 are believed to be accurate to two significant figures. For comparison, some points computed from Rice's approximate formula for $f(y)$ (equation 3.9–20 of [Ref. 1]) are shown on Fig. 3. Rice's formula is seen to fit the tails of $f(y)$ well for large y, but the central portion of the distribution is given accurately only for large values of τ. However, the approximate cumulative distribution obtained by integrating Rice's formula agrees quite well with $F(y)$ for a wide range of τ values as is seen in Fig. 6.

The approximation in question assumes a χ^2 type distribution

$$\tilde{f}(y) = \frac{y^{(n/2)-1}e^{-(y/2q^2)}}{(2q^2)^{n/2}\Gamma\left(\frac{n}{2}\right)}.$$

The parameters q and n are chosen to make the first two moments of this density agree with the true first two moments of y. That is, for the normalization $Ey = 1$ adopted here, the equations $q^2n = 1$ and $2nq^4 = \sigma^2$ serve to determine q and n. These formulae give $n = 2/\sigma^2$. Since for large τ, $\sigma^2 \sim 2/\tau$ for bandpass noise, $n \sim \tau = 2(2\delta)T$. That is, for large τ, the bandpass noise acts like a χ^2 variate with $2(2\delta)T$ degrees of freedom in agreement with an argument easily derived from the sampling theorem.

III. GENERAL THEORY

Let $N(t)$ be a Gaussian noise with mean zero and covariance

$$\rho(t, t') = E[N(t)N(t')]$$

where as usual E denotes expectation. In studying properties of $N(t)$ in a finite time interval, say $(-T/2, T/2)$, it is convenient to make an expansion in terms of an orthonormal set of functions, $\varphi_n(t)$, $n = 0, 1, 2, \ldots$. We write

$$N(t) = \sum_0^\infty n_i\varphi_i(t), \qquad |t| \leqq \frac{T}{2}$$

where

$$n_i = \int_{-T/2}^{T/2} N(t)\varphi_i(t)\,dt, \qquad i = 0, 1, 2, \cdots$$

and

$$\int_{-T/2}^{T/2} \varphi_i(t)\varphi_j(t)\,dt = \delta_{ij}, \qquad i, j = 0, 1, 2, \cdots.$$

As is well known,[2] it is particularly convenient in this description of the noise to choose as the orthonormal set, φ_i, the solutions of the homogeneous Fredholm equation with $\rho(t, t')$ as kernel. That is, the φ's are chosen so that

$$\lambda_i\varphi_i(t) = \int_{-T/2}^{T/2} \rho(t, t')\varphi_i(t')\,dt', \qquad |t| \leqq \frac{T}{2}, \qquad i = 0, 1, 2, \cdots. \tag{4}$$

For, with this choice of the φ's, it is easily shown that the n_i are *independent* Gaussian variates with mean zero and variance $E(n_i^2) = \lambda_i$, $i = 0, 1, 2, \ldots$. We assume in all that follows that the λ's are so labelled that $\lambda_0 \geqq \lambda_1 \geqq \lambda_2 \geqq \ldots$.

Consider now the average power, y, of a finite sample of the noise. It follows that

$$\begin{aligned} y &= \frac{1}{T} \int_{-T/2}^{T/2} N^2(t)\, dt = \frac{1}{T} \sum_0^\infty n_i^2 \\ &= \sum_0^\infty a_i x_i^2, \end{aligned} \tag{5}$$

where

$$x_i = \frac{n_i}{\sqrt{\lambda_i}}$$

and

$$a_i = \frac{\lambda_i}{T}. \tag{6}$$

Equation (5) exhibits y as a linear combination of independent random variables. The x_i are independent Gaussian variables all with mean zero and variance unity. The characteristic function, $C(\xi)$, for y then follows readily. One has

$$\begin{aligned} C(\xi) &= Ee^{i\xi y} = Ee^{i\xi \Sigma a_j x_j^2} = \prod_{j=0}^\infty Ee^{i\xi a_j x_j^2} \\ &= \prod_{j=0}^\infty (1 - 2i\xi a_j)^{-1/2}. \end{aligned} \tag{7}$$

Here, as throughout this paper, the positive square root of a complex quantity is taken to have an angle between $-(\pi/2)$ and $+(\pi/2)$ radians (the cut line is along the negative real axis).

From the characteristic function (7), the semi-invariants of y can be calculated. By definition[3] of the semi-invariants, κ_j,

$$\log C(\xi) = \sum_1^\infty \frac{\kappa_\nu}{\nu!} (i\xi)^\nu.$$

From (7) and the expansion

$$\log (1 - x) = -\sum_{n=1}^\infty \frac{x^n}{n},$$

it follows that

$$\log C(\xi) = -\frac{1}{2}\sum_{j=0}^{\infty} \log (1 - 2i\xi a_j) = \frac{1}{2}\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} \frac{(2i\xi a_j)^k}{k}$$

$$= \sum_{k=1}^{\infty} \frac{(i\xi)^k}{k!} \kappa_k$$

where

$$\kappa_k = (k - 1)!\, 2^{k-1} \textstyle\sum_{j=0}^{\infty} a_j^{\,k}. \tag{8}$$

From the semi-invariants, the moments of y can be found as in Reference 3.

The formula (8) for the semi-invariants can be put in a convenient form not involving the a_j explicitly. From the well known expansion[4]

$$\rho(t, t') = \textstyle\sum_0^\infty \lambda_j \varphi_j(t)\varphi_j(t')$$

and the orthonormal properties of the φ's, one finds

$$\kappa_k = \frac{(k - 1)!\, 2^{k-1}}{T^k} \int_{-T/2}^{T/2} \rho^{(k)}\ (t, t)\ dt, \tag{9}$$

where the iterated kernel $\rho^{(k)}(t, t')$ is defined by

$$\rho^{(1)}(t, t') = \rho(t, t'),$$

$$\rho^{(n)}(t, t') = \int_{-T/2}^{T/2} \rho(t, x)\rho^{(n-1)}(x, t')\ dx, \qquad n = 2, 3, \cdots.$$

The determination of the higher order iterated kernels generally becomes difficult in practice.

The expression (9) is of the form conjectured by Rice[1] on the basis of computing the first four semi-invariants of y. The formula (7) was given by Kac and Siegert[2] and (9) was noted by Arthur[5] in a special case in connection with the analysis of a frequency discriminator.

The probability density function for y is obtained as the Fourier transform of $C(\xi)$,

$$f(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\xi y}\, d\xi}{\prod_{j=0}^{\infty} (1 - 2i\xi a_j)^{1/2}} \tag{10}$$

and the cumulative distribution function can be written as

$$F(y) = 1 - \int_y^\infty f(x)\ dx. \tag{11}$$

Much of the remainder of this paper will be concerned with evaluating (10) and (11) for specific noises.

IV. COMPUTATIONAL FORM FOR $f(y)$

The evaluation of the integral (10) presents many difficulties even with modern computing machinery. From the physical origins of the problem under discussion, it is clear that for small values of T, $f(y)$ must be a rather broad function (non-localized), whereas for large values of T it must approach a δ-function centered at the point $y = \rho(0, 0)$ when the noise is assumed ergodic. The behavior of (10) therefore depends in detail on the manner in which the a_j approach zero with increasing j.

One seemingly attractive approach to the problem is to truncate the sum in (5) at $i = M$ and correspondingly obtain a product with j running from 0 to M in the denominator of the integral in (10). Procedures are described in the literature[6, 7] for computing the distribution of a finite quadratic form in Gaussian variables. Estimates of the error due to truncation can also be obtained rather readily. Unfortunately, the best such estimates obtained by the author showed that for small values of β or τ, M must be taken quite large (50 or 60) to obtain answers guaranteed accurate to two decimal places. Furthermore, the convergence of the computational schemes described[6, 7] turned out to be very slow. The

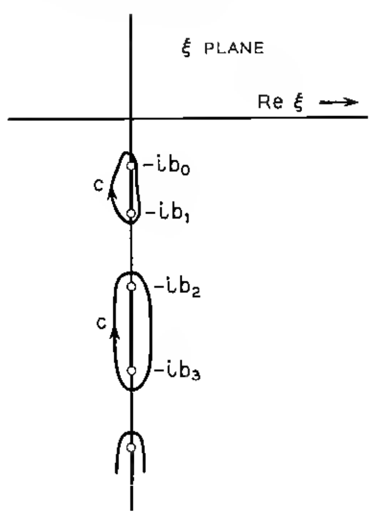

Fig. 12 — Cut lines and contour in complex ξ plane.

following alternative approach which can also be applied to finite sums of the form (5) was used to obtain the curves presented here.

The function $\Pi(1 - 2i\xi a_j)^{1/2}$ in the denominator of (10) has branch points at $-ib_j$, where

$$b_j = \frac{1}{2a_j} = \frac{T}{2\lambda_j}, \qquad j = 0, 1, 2, \ldots. \quad (12)$$

The b's are real positive quantities, for the λ's are eigenvalues of a real symmetric positive definite kernel and must be real positive numbers. Line segment cut-lines are inserted in the complex ξ-plane from $-ib_{2j}$ to $-ib_{2j+1}$, $j = 0, 1, 2, \ldots$ as shown in Fig. 12. When $y < 0$, the value of (10) is zero as can be seen by closing the contour in the upper half plane. When $y \geqq 0$, the contour of integration in (10) is displaced from the axis of reals to the contour, C, shown in Fig. 12. This displacement of contour is easily justified if $\Pi(1 - 2i\xi a_j)^{1/2}$ is of exponential order less than unity, a condition which will be fulfilled in the examples to be treated. The change of variable $\zeta = i\xi$ rotates the contour of Fig. 12 by 90° in the positive direction. If one now collapses the closed contour curves about the cut-lines and takes proper care of the convention already set forth for the square root sign, there results,

$$f(y) = \textstyle\sum_{k=0}^{\infty}(-1)^k I_k$$

where

$$I_k = \frac{1}{\pi}\int_{b_{2k}}^{b_{2k+1}} \frac{e^{-yt}\,dt}{\sqrt{|D(t)|}}, \qquad k = 0, 1, 2, \ldots$$

and where

$$D(t) = \prod_{j=0}^{\infty}\left(1 - \frac{t}{b_j}\right). \quad (13)$$

$D(t)$ is closely related to the Fredholm determinant (Reference 4, Chapter 11) of ρ.

In the application to be treated below,

$$D(t) = H(z) \quad (14)$$

where

$$z = g(t) \quad (15)$$

is a non-negative monotone increasing real function of t for $t \geqq b_0$. Denote its inverse by $t = h(z)$. Let

$$z_k = g(b_k)$$

$$c_k = \tfrac{1}{2}(z_{2k+1} - z_{2k})$$

$$d_k = \tfrac{1}{2}(z_{2k+1} + z_{2k})$$

for $k = 0, 1, 2, \ldots$ and let

$$z = c_k \cos \pi x + d_k .$$

Then straightforward substitution yields

$$I_k = \int_0^1 \frac{e^{-yh(z)} h'(z)\, dx}{\sqrt{\left| \dfrac{H(z)}{(z - z_{2k})(z_{2k+1} - z)} \right|}}, \qquad k = 0, 1, 2, \ldots, \tag{16}$$

$$f(y) = \sum_{k=0}^{\infty} (-1)^k I_k . \tag{17}$$

Similarly, one obtains

$$F(y) = 1 - \sum_{k=0}^{\infty} (-1)^k J_k \tag{18}$$

with

$$J_k = \int_0^1 \frac{e^{-yh(z)} h'(z)\, dx}{h(z) \sqrt{\left| \dfrac{H(z)}{(z - z_{2k})(z_{2k+1} - z)} \right|}}. \tag{19}$$

Equations (16) to (19) were used to compute the curves discussed in Section II. The denominators of the integrals in (16) and (19) have no zeros in the range of integration. By use of Gauss's method of numerical integration,[8] evaluation of the integral at $x = 0$ and $x = 1$ where the denominator is an indeterminate form was avoided. In the applications made, it can be shown that for sufficiently large k, I_k and J_k decrease monotonely. Since the series (17) and (18) are alternating, an estimate of the error made by terminating the series at a finite value of k can be obtained. In all cases computed, it was never necessary to take k larger than 18, to obtain 1 per cent accuracy in the final result.

V. DETERMINATION OF EIGENVALUES AND $H(z)$*

For stationary processes, the kernel of the integral equation (4) becomes a difference kernel; that is, $\rho(t, t') = \rho(t - t')$ where $\rho(x)$ is a positive definite function. The Fourier transform of ρ, namely

$$w(f) = \int_{-\infty}^{\infty} e^{2\pi i f x} \rho(x)\, dx$$

is non-negative and is the power density spectrum of the processes.

* An alternative method of evaluating $H(z)$ is described in Reference 12.

Analytic solutions to the integral equation (4) are known in this case only for a relatively small class of kernels. Fortunately, this class is one of considerable interest in communication applications. It is the class of ρ whose spectra $w(f)$ are rational functions of f^2; i.e., ratios of polynomials in f^2. Such spectra are obtained by passing white noise through a finite passive physical electrical network with lumped constants. Details of the method of solution are given in References 9 and 10. It must be pointed out that, even in this case, solutions can be carried out practically only for polynomials of small degree.

5.1 *RC Noise*

If white Gaussian noise is applied to a series RC circuit, the voltage across the capacitor has a power density spectrum proportional to

$$w(f) = \frac{2\alpha}{\alpha^2 + 4\pi^2 f^2} \tag{20}$$

where $\alpha = 1/RC$ is the nominal cut-off frequency of the circuit. The covariance function corresponding to (20) is

$$\rho(t) = e^{-\alpha|\tau|}. \tag{21}$$

Solutions to (4) with this kernel are given in detail in both References 9 and 10.

Let

$$\beta = \frac{\alpha T}{2}. \tag{22}$$

Then

$$b_k = \frac{1}{2\beta}[\beta^2 + z_k^2], \qquad k = 0, 1, 2, \ldots, \tag{23}$$

where the z_k are non-negative roots of either of the equations

$$z \tan z = \beta \tag{24}$$

$$z \cot z = -\beta. \tag{25}$$

If the z_k are labelled so that $z_0 < z_1 < z_2 < \ldots$, then it is readily seen that $z_k \sim k(\pi/2)$, so that $b_k \sim k^2(\pi^2/8\beta)$. The convergence exponent (see Reference 11, p. 14) of the sequence b_k is therefore $\frac{1}{2}$. It follows then (Reference 11, 2.6.5, p. 19) that $D(t)$ as given by (13) is an entire function of order $\frac{1}{2}$.

Now the function $(e^{-2\beta}/\beta)[\beta \cos z + z \sin z]\,[\cos z + \beta(\sin z/z)]$, where $z = \sqrt{2\beta t - \beta^2}$, is an entire function of t of order $\frac{1}{2}$. Its only zeros are

at the points $t = b_k$, $k = 0, 1, 2, \ldots$, given by (23), (24) and (25). At $t = 0$, it has the value unity. It is, therefore, equal to $D(t)$ as can be seen from Hadamard's Factorization Theorem (Reference 11, 2.7.1, p. 22).

The quantities necessary to evaluate (16) and (19) are therefore all known for this case:

$$H(z) = \frac{e^{-2\beta}}{\beta} [\beta \cos z - z \sin z] \left[\cos z + \beta \frac{\sin z}{z} \right]$$

$$h(z) = \frac{1}{2\beta} [\beta^2 + z^2]$$

and the z_k are given by the positive roots of (24) and (25).

The first two semi-invariants of y are found to be

$$\kappa_1 = Ey = 1,$$

$$\kappa_2 = E(y - 1)^2 = \sigma^2 = \frac{1}{4\beta^2} [4\beta - 1 + e^{-4\beta}].$$

5.2 *RLC Noise*

If white Gaussian noise is applied to a series RLC circuit, the voltage across the resistor has a power density spectrum proportional to

$$w(f) = \frac{2Q}{\omega_0} \frac{\omega^2}{\omega^2 + \left(\frac{Q}{\omega_0}\right)^2 (\omega^2 - \omega_0^2)^2} \tag{26}$$

where $\omega = 2\pi f$, $Q = \omega_0 L/R$ and $\omega_0^2 = 1/LC$. Introducing parameters u and v defined by

$$u^2 + v^2 = \omega_0^2 \left[\frac{1}{Q^2} - 2 \right]$$

$$uv = \omega_0^2, \qquad \operatorname{Re} u \geqq 0, \qquad \operatorname{Re} v \geqq 0,$$

one finds

$$w(f) = \frac{2(u + v)\omega^2}{(u^2 + \omega^2)(v^2 + \omega^2)}$$

and

$$\rho(\tau) = \frac{1}{u - v} [ue^{-u|\tau|} - ve^{-v|\tau|}]. \tag{27}$$

In the special case $Q = \frac{1}{2}$,

$$\rho(\tau) = (1 - \omega_0 | \tau |)e_0^{-\omega_0 |\tau|}. \tag{28}$$

Solution of (4) with (27) or (28) as kernel is relatively straightforward by the methods of References 9 and 10, although quite laborious. Details can be found in Appendix 2.

Suppose Q and ω_0 are positive real quantities. Then the eigenvalues $\lambda_k = T/2b_k$ are given by

$$b_k = \frac{1}{2r} [r^2 + z_k^2], \qquad k = 0, 1, 2, \ldots \tag{29}$$

where the z_k are non-negative roots of either

$$2r \cos z = (z^2 - r^2) \frac{\sin z}{z} + (z^2 + r^2) \frac{\sin \sqrt{z^2 + s^2}}{\sqrt{z^2 + s^2}} \tag{30}$$

or

$$2r \cos z = (z^2 - r^2) \frac{\sin z}{z} - (z^2 + r^2) \frac{\sin \sqrt{z^2 + s^2}}{\sqrt{z^2 + s^2}}. \tag{31}$$

Here

$$r = \frac{\omega_0 T}{2Q}, \qquad s = \omega_0 T. \tag{32}$$

The eigenfunctions belonging to roots of (30) are of the form

$$A_k \cos \tfrac{1}{2}(z_k + \sqrt{z_k^2 + s^2})t + B_k \cos \tfrac{1}{2}(z_k - \sqrt{z_k^2 + s^2})t$$

while those belonging to roots of (31) are of the form

$$C_k \sin \tfrac{1}{2}(z_k + \sqrt{z_k^2 + s^2})t + D_k \sin \tfrac{1}{2}(z_k - \sqrt{z_k^2 + s^2})t.$$

It is interesting to note that when the λ's are ordered in the usual way, the corresponding eigenfunctions do not in general alternate between even and odd functions of t.

The infinite product (13) with the b's given by (29), (30) and (31) can be written in closed form by arguments similar to those used in Section 5.1. From (30) and (31), it is seen that asymptotically successive z_k are separated by $\pi/2$, so that b_k grows like k^2 and one is again dealing with an entire function of order $\frac{1}{2}$.* For the pertinent quantities

* More generally, it can be shown that for rational spectra if $w(f) \sim f^{-2p}$ then $\lambda_n \sim n^{-2p}$. (Private communication to author by A. Beurling.)

of (16) and (19), one finds

$$H(z) = \frac{e^{-2r}}{4r^2}\left[(z^2 - r^2)\frac{\sin z}{z} - 2r\cos z + (z^2 + r^2)\frac{\sin\sqrt{z^2+s^2}}{\sqrt{z^2+s^2}}\right] \cdot \left[(z^2 - r^2)\frac{\sin z}{z} - 2r\cos z - (z^2 + r^2)\frac{\sin\sqrt{z^2+s^2}}{\sqrt{z^2+s^2}}\right], \tag{34}$$

$$h(z) = \frac{1}{2r}[z^2 + r^2] \tag{35}$$

with the z_k given as roots of (30) and (31).

The first two semi-invariants of y are

$$u_1 = Ey = 1$$

$$u_2 = \sigma^2 = \frac{1}{2r^2}\left[2r - 1 + e^{-2r} + 2\frac{r^2e^{-2r}}{r^2 - s^2}\sinh^2\sqrt{r^2 - s^2}\right].$$

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APPENDIX I

Let $N(t) = a\cos\omega_0 t + b\sin\omega_0 t$, where a and b are independent Gaussian variates with mean zero and variance unity. Then y as defined by (1) is obtained by direct integration as

$$y = \alpha a^2 + \beta b^2$$

where

$$\alpha = \frac{1}{2}\left(1 + \frac{\sin s}{s}\right), \qquad \beta = \frac{1}{2}\left(1 - \frac{\sin s}{s}\right), \qquad s = \omega_0 T.$$

Since y is the sum of independent random χ^2 variables, the density for y can be obtained as the convolution

$$f(y) = \int_0^y \frac{e^{-(x/2\alpha)}}{\sqrt{2\pi\alpha x}}\frac{e^{-((y-x)/2\beta)}}{\sqrt{2\pi\beta(y-x)}}\,dx.$$

The substitution $x = (y/2)(1 + \cos\theta)$ in this integral leads to

$$f(y) = \frac{e^{-(y/4)[(1/\alpha)+(1/\beta)]}}{2\sqrt{\alpha\beta}} \frac{1}{\pi} \int_0^\pi e^{(y/4)(\alpha^{-1}-\beta^{-1}) \cos\theta} \, d\theta$$

$$= \frac{e^{-(y/4)[(1/\alpha)+(1/\beta)]}}{2\sqrt{\alpha\beta}} \qquad J_0\left[i\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)\frac{y}{4}\right].$$

Finally, if $\sin\varphi = \sin s/s$,

$$f(y) = \sec\varphi e^{-y\sec^2\varphi} J_0(iy \tan\varphi \sec\varphi).$$

APPENDIX 2

The power spectrum corresponding to the covariance (27) can be written as

$$w = -\frac{2(u+v)p^2}{(p^2 - u^2)(p^2 - v^2)}$$

where $p = i\omega = 2\pi i f$. From Reference 9, then, solutions to (4) with the kernel (27) must satisfy the differential equation

$$\left(\frac{d^2}{dt^2} - u^2\right)\left(\frac{d^2}{dt^2} - v^2\right)\varphi(t) = -\frac{2(u+v)}{\lambda}\frac{d^2}{dt^2}\varphi(t)$$

or

$$\left(\frac{d^2}{dt^2} - \alpha^2\right)\left(\frac{d^2}{dt^2} - \beta^2\right)\varphi(t) = 0, \tag{i}$$

where

$$\alpha^2 + \beta^2 = u^2 + v^2 - \frac{2(u+v)}{\lambda},$$

$$\alpha^2\beta^2 = u^2v^2 = \omega_0^4.$$

We choose α and β so that Re $\alpha \geqq 0$, Re $\beta \geqq 0$. If $\alpha \neq \beta$, then φ is a linear combination of the elementary functions $e^{\alpha t}$, $e^{-\alpha t}$, $e^{\beta t}$, $e^{-\beta t}$.

It is easy to verify that if φ is a solution to (4) with a kernel $\rho(t, t') = \rho(|t - t'|)$, then $\varphi(t) + \varphi(-t)$ and $\varphi(t) - \varphi(-t)$ are also solutions. We can, therefore, restrict attention to even and odd solutions of (4). On substituting

$$\varphi(t) = A \cosh \alpha t + B \cosh \beta t$$

into (4), one finds

$$\frac{\lambda}{T} = -\frac{\alpha^2 r}{(u^2 - \alpha^2)(v^2 - \alpha^2)} = -\frac{\beta^2 r}{(u^2 - \beta^2)(v^2 - \beta^2)}, \tag{ii}$$

$$
\begin{aligned}
&A\,\frac{u\cosh\dfrac{\alpha T}{2}+\alpha\sinh\dfrac{\alpha T}{2}}{u^2-\alpha^2}+B\,\frac{u\cosh\dfrac{\beta T}{2}+\beta\sinh\dfrac{\beta T}{2}}{u^2-\beta^2}=0,\\
&A\,\frac{v\cosh\dfrac{\alpha T}{2}+\alpha\sinh\dfrac{\alpha T}{2}}{v^2-\alpha^2}+B\,\frac{v\cosh\dfrac{\beta T}{2}+\beta\sinh\dfrac{\beta T}{2}}{v^2-\beta^2}=0.
\end{aligned}
\tag{iii}
$$

The determinant of the system (iii) must vanish. A bit of algebra shows this to be equivalent to

$$
2r\cosh x+(x^2+r^2)\frac{\sinh x}{x}+(x^2-r^2)\frac{\sinh\sqrt{x^2-s^2}}{\sqrt{x^2-s^2}}=0\,, \tag{iv}
$$

where $x=(\alpha+\beta)(T/2)$. It is not difficult to show that for positive ω_0 and Q, this equation has roots only if α and β, and hence x, are pure imaginary. Writing $x = iz$, (iv) become (30) and (ii) yields (29).

The substitution of

$$
\varphi(t) = C\sinh\alpha t + D\sinh\beta t
$$

into (4) again yields (ii) and equations analogous to (iii) with sinh and cosh interchanged. A similar analysis then gives (31).

If $\alpha = \beta$, then from (i), φ must be of the form $A\cosh\alpha t + Bt\sinh\alpha t$ or $C\sinh\alpha t + Dt\cosh\alpha t$. Substitution of these forms into (4) yields equations which cannot be satisfied for positive ω_0 and Q except by the trivial solution $A = B = C = D = 0$.

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